Geometric Properties and Non-blowup of 3-D Incompressible Euler Flow *

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Abstract

By exploring a local geometric property of the vorticity field along a vortex filament, we establish a sharp relationship between the geometric properties of the vorticity field and the maximum vortex stretching. This new understanding leads to an improved result of the global existence of the 3-D Euler equation under mild assumptions that are consistent with the observations from recent numerical computations.

1 Introduction

The problem of global existence/blowup of smooth solutions for the three-dimensional incompressible Euler flow, which is governed by the 3-D Euler equations:

$$\begin{aligned} u_t + (u \cdot \nabla)u &= \nabla p \\ \nabla \cdot u &= 0 \\ u\,|_{t=0} &= u_0 \end{aligned} \tag{1.1}$$

is a long time outstanding question. It plays a very important role in understanding the core problems in hydrodynamics such as the onset of turbulence. Much effort has been made in both theory and numerics trying to answer this question, see, e.g. Beale-Kato-Majda [2], Caflisch [3], Constantin-Fefferman-Majda [6], and Babin-Mahalov-Nicolaenko [1]. Through these efforts, it is realized that the above issue is

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closely related to the stretching of the vorticity $\omega \equiv \nabla \times u$, which is governed by the following evolution equation:

$$\begin{aligned} \frac{D\omega}{Dt} &= (\nabla u)\omega \\ \omega\,|_{t=0} &= \omega_0 = \nabla \times u_0, \end{aligned} \tag{1.2}$$

where $\frac{D}{Dt} \equiv \partial_t + (u \cdot \nabla)$ is the material derivative. In a well-known paper by Beale, Kato and Majda ([2]), it has been shown that the smooth solution $u(x,t)$ for the 3-D Euler flow blows up at $t = T$ if and only if $\int_0^t \|\omega(\cdot, s)\|_\infty ds \nearrow \infty$ as $t \nearrow T$. Some variants and improvements have appeared in the last two decades. Very recently, Ogawa and Taniuchi [13] have shown that the above L^∞ norm estimate on vorticity in Beale-Kato-Majda's blow-up criterion can be replaced by a weaker BMO norm estimate.

The above result implies that we should study the dynamic growth of vorticity in the flow. It has been observed from the early 80s in the last century that small vortex tubes dominate the vorticity field in later times of the flow, especially in near-singular situations. This observation gives impetus on studying the details of the evolution of regions in which vorticity concentrates. People have been trying to find conditions on the geometry of the vorticity field that can be used to exclude blow-up by rigorous mathematical proofs. In particular, Constantin, Fefferman and Majda [6] prove that if there is up to time T an $O(1)$ region in which the vorticity vector $\xi(x,t) \equiv \frac{\omega(x,t)}{|\omega(x,t)|}$ is smoothly directed, i.e., the maximum norm of $\nabla \xi$ in this region is L^2 integrable in time from 0 to T, and the maximum norm of velocity in some $O(1)$ neighborhood of this region is uniformly bounded in time, then no blow-up can occur in this region up to time T. Another way of attacking the problem is by taking advantage of incompressibility and is explored by Cordoba and Fefferman in [7], in which the possibility of uniform collapsing of vortex tubes with $O(1)$ length that don't twist or bend violently are ruled out under the assumption that the infinity norm of velocity in a neighborhood of the region under consideration is integrable in time.

Many numerical computations have been performed to search for possible candidates for a finite time blow-up. Some of the better known examples are given by Kerr [9, 10, 11, 12], Pelz [14, 15], and Grauer-Marliani-Germaschewski [8]. Up to now, the most probable candidate is the anti-parallel vortex tube setting, which has been carefully studied by Kerr and others taking advantage of the ever-growing computing power (see e.g. [9, 10, 11, 12]). The magnitudes of maximum vorticity observed in all of these numerical computations can be fitted by a growth rate of $(T-t)^{-1}$ which is the critical case in the Beale-Kato-Majda blow-up criterion. Although many numerical results suggesting finite time blow-ups have been obtained, no conclusive claim has been drawn so far. One thing that is worth mentioning is that, in all these computations, it

is observed that vorticity is concentrated in small regions that are shrinking with time, and the shrinkage rate is related to some inverse power of the maximum vorticity.

There is still little overlap of the cases studied by the theoretical and numerical groups, although many results have been obtained and efforts made. All the existing theorems deal with $O(1)$ regions in which the vorticity vector is assumed to have some regularity, while in numerical computations, the regions which have such regularity and contain maximum vorticity are all shrinking with time. In this article, we try to narrow this gap by considering cases that are consistent with the numerical observations. We prove that no finite time blow-up can occur if some mild assumptions on the geometric properties of the vorticity vector and the behavior of the velocity field are satisfied.

The key to our analysis is the following understanding. The magnitudes of vorticity at any two points on one vortex line are related to each other by the geometry of the vorticity field through the incompressibility condition. This understanding has not been seen in the existing literature and is a key to our analysis. Another key factor to our work is the reformulation of the problem into a vortex filament setting. Unlike previous vorticity growth formulas, this formulation reveals the anisotropic nature of vortex stretching and enables us to obtain an improved global existence result for the incompressible 3-D Euler equation.

Specifically, we obtain two results. The first one says that if the divergence of the vorticity vector, $\nabla \cdot \xi$, along the vortex line segment $[s_1, s_2]$ containing the point of maximum vorticity is integrable, i.e.,

$$\left| \int_{s_1}^{s_2} (\nabla \cdot \xi) ds \right| \leq C(T), \quad 0 \leq t \leq T, \tag{1.3}$$

where s is the arc length variable, and $s_2 > s_1$, then no point singularity is possible up to time T. If the vorticity blows up at one point within this vortex line segment, then the vorticity must blow up simultaneously *at the same rate* in the *entire* vortex line segment.

The usefulness of the first result lies in the fact that the weakly regular orientedness condition expressed by (1.3) is extremely localized. It is a condition along one local vortex line segment. Since people have observed numerically some form of partial regularity of the vorticity vector in a small inner region containing the point of maximum vorticity, we can readily apply this criterion to check the validity of some singularity scenarios reported in numerical computations. For example, in Pelz's computation [14, 15], a tube-shaped region of length scale $(T-t)^{1/2}$ is highlighted as a candidate for a finite time blow-up. A simple calculation shows that the criterion (1.3) is satisfied within this inner tube-shaped region. This casts doubt on the validity of Pelz's claim on the finite time formation of a point singularity. To validate Pelz's claim, one needs to perform a more careful numerical study to check whether there exists a nonvanishing vortex line segment within which condition (1.3) is satisfied or whether the vorticity

within the inner tube-shaped region blows up at the same rate.

Our second theorem proves the global existence of the incompressible 3-D Euler equation under some relatively mild assumptions. In this theorem, we deal with the case when the length of the weakly regularly oriented vortex line segment can shrink to zero as the time approaches to the alleged singularity time. It gives a sharper dynamic description of the vortex stretching. Assume that at each time t there exists some vortex line segment L_t on which the maximum vorticity is comparable to the global maximum vorticity. We denote by $L(t)$ the arc length of this vortex line segment. In addition to satisfying a variant of (1.3), we assume that $L(t)\|\kappa\|_{L^\infty(L_t)}$ (here κ is curvature of the vortex line L_t) is bounded, and that the maximum norms of the normal and tangential velocity components along the local vortex line segment L_t are integrable in time. The length of the local vortex line segment, $L(t)$, is allowed to shrink to zero as time approaches to the alleged singularity time. Under these assumptions, we can prove that no finite time blow-up is possible. To simplify our analysis and to obtain a concrete rate of shrinkage of $L(t)$, we present a slightly weaker version of the result in this paper by assuming an upper bound of the growth rate in time of the normal and tangential velocity components along L_t.

Our second theorem to some extent improves the previous results obtained by Constantin-Fefferman-Majda [6] and Cordoba-Fefferman [7]. First of all, our result requires a very localized and weaker assumption on the regularity of the vorticity vector ξ. In [6], the gradient of the vorticity vector is assumed to be L^2 integrable in time in an $O(1)$ region containing the maximum vorticity. In contrast, we only assume that the divergence of the vorticity vector is integrable along a local vortex line segment and $L(t)\|\kappa\|_{L^\infty(L_t)}$ is bounded. The length of the vortex line segment, $L(t)$, can shrink to zero as the time approaches to the alleged singularity time. The numerical computations by Kerr [9] and Pelz [14] have demonstrated that there is indeed a small region in which vorticity attains its global maximum and the vorticity vector has some partial regularity. However, the size of this region shrinks rapidly to zero in a rate proportional to some inverse power of maximum vorticity. Thus there is a significant gap between the assumption on the smoothly directed region in [6] and what has been observed numerically. On the other hand, our assumption on the partial regularity of the vorticity vector is very mild and localized along one vortex line segment so that we can apply our result to check the validity of some numerical studies, such as those by Kerr ([9, 10, 11, 12]), in which finite time singularity of the 3-D Euler equation has been alleged.

It is also worth mentioning that in our second theorem, we only assume that the normal and tangential velocity components within the local vortex line segment L_t are integrable in time. In comparison, the maximum norm of the entire velocity field within an $O(1)$ region is assumed to be bounded in [6]. In the case of the collapse of a regular vortex tube, the maximum velocity is given by the rotational component of the velocity

field in the cross section normal to the direction of the vortex tube. As the vortex tube collapses, the rotational component of the velocity field may blow up proportional to the square root of the maximum vorticity from Kelvin's circulation theorem. The normal velocity component generally corresponds to the speed of the motion of the vortex tube, which may remain bounded even in the collapse of the vortex tube. On the other hand, we expect that the tangential velocity component is smaller than the maximum velocity field due to the cancellation of vorticity vectors in the inner region, leading to one order reduction of the velocity kernel.

We would like to emphasize that our analysis reveals a close connection between the global existence of the 3-D Euler equation and the local geometric property of a vortex line segment containing the maximum vorticity. This observation sheds useful light in our future effort in studying the dynamical interplay between the local geometric property of the vortex filament and the maximum vortex stretching.

This paper is organized as follows. We highlight our main results in Section 2 and describe their implications by applying them to study some recent numerical computations. In Section 3, we explore the geometry of the vorticity field and the incompressibility condition in depth and prove our two main theorems.

Notations

Throughout this paper, we will reserve some characters for some particular quantities according to the following rules of notations:

- C or c: generic constants, whose value may change from line to line. When not otherwise indicated, the values of $C(c)$ are independent of any of the data.
- ξ is always the direction of vorticity vectors, i.e., $\xi \equiv \omega/|\omega|$.
- T will always denote the alleged time when a finite time blow-up occurs.

We will also use the following notations for convenience:

- $\sim$: We write $a(t) \sim b(t)$ if there are absolute constants $c, C > 0$ such that $c\,|a(t)| \le |b(t)| \le C\,|a(t)|$.
- $\gtrsim (\lesssim)$: We write $a(t) \gtrsim b(t)$ if there is an absolute constant $c > 0$ such that $|a(t)| \ge c\,|b(t)|$. $a(t) \lesssim b(t)$ is defined similarly.

2 Main Results and Their Implications

In this section, we present our two main results. The first one says that if the divergence of the vorticity vector along a nonvanishing local vortex line segment containing the maximum vorticity is integrable in time, then no point singularity is possible. If the vorticity blows up at one point, then the vorticity along this vortex line segment must

blow up simultaneously at the singularity time. Our second result gives a sharp criterion for the dynamic blow-up of vorticity. With additional assumptions on the curvature of the local vortex line and the growth rate of the normal and tangential velocity components along the vortex line, we prove that no blowup is possible in finite time. Below we describe these two results and discuss how they can be applied to check validity of some numerical studies in which singularities of the 3-D Euler equation have been alleged.

Our first theorem is as follows:

Theorem 1. *We consider any 3-D incompressible flow (Euler or Navier-Stokes). Let* $x(t)$ *be a family of points such that* $|\omega(x(t),t)| \gtrsim \Omega(t) \equiv \|\omega(\cdot,t)\|_{L^\infty(\mathbf{R}^3)}$. *Assume that for all* $t \in [0,T)$ *there is another point* $y(t)$ *on the same vortex line as* $x(t)$*, such that the direction of vorticity* $\xi(x,t) \equiv \frac{\omega(x,t)}{|\omega(x,t)|}$ *along the vortex line between* $x(t)$ *and* $y(t)$ *is well-defined. If we further assume that*

$$\left| \int_{x(t)}^{y(t)} (\nabla \cdot \xi)(s,t)\ ds \right| \leq C \tag{2.1}$$

for some absolute constant C*, and*

$$\int_0^T |\omega(y(t),t)|\ dt < \infty,$$

then there will be no blow-up up to time T*. Moreover, we have*

$$e^{-C} \leq \frac{|\omega(x,t)|}{|\omega(y,t)|} \leq e^{C}. \tag{2.2}$$

The proof of Theorem 1 is quite simple and will be deferred to Section 3.

The above theorem gives a practical criterion on judging possible blow-up in a numerical computation. It also suggests that, when searching for a finite time blow-up numerically, one has to pay attention to the geometric property of vortex filaments. It is not enough to just track the maximum vorticity magnitude and the point at which this maximum is attained. The vorticity magnitudes at other points are also crucial. In particular, the above theorem implies that if there is a nonvanishing vortex line segment containing the maximum vorticity up to time T such that (2.1) is satisfied, then no point singularity is possible up to this time T. To illustrate, we apply Theorem 1 to the numerical results of Pelz [14, 15].

Example 1. In [14, 15], Pelz studied a class of incompressible flows with strong symmetry and conjectured that such flows can lead to a finite time blow-up. In these computations, vorticity is concentrated in small vortex tubes of length scale $\sim (T-t)^{1/2}$. After a re-scaling $x \mapsto (T-t)^{-1/2}x$, these tubes seem to have a regular shape. This

suggests that the length of this inner region scales like $(T-t)^{1/2}$ and the scaling of $\nabla \cdot \xi$ within this inner region is of the order $(T-t)^{-1/2}$. Let us take the point $x(t)$ to be the point inside one tube where the maximum vorticity is attained, and $y(t)$ to be a point on the same vortex line, but outside the tube. It is easy to check that within this inner region, condition (2.1) is satisfied. By Theorem 1 we see that if the maximum vorticity *outside* these small tubes is integrable in time, then there is no blow-up inside the tubes. It is likely that the maximum vorticity outside these small tubes has a growth rate smaller than that inside these small regions. This casts doubt on the validity of Pelz's claim on the finite time formation of a point singularity. To validate Pelz's claim, one needs to perform more careful numerical study to check whether there exists a nonvanishing vortex line segment within which condition (1.3) is satisfied or whether the vorticity within the inner tube-shaped region blows up at the same rate.

Our second result is concerned with the dynamic blow-up of one vortex line. As in [2], we assume that the initial velocity field, u_0, is smooth and vanishes rapidly at infinity, more specifically $u_0 \in H_0^{7/2}(\mathbb{R}^3)$. Denote by $\Omega(t)$ the maximum vorticity in the whole 3-D space. We consider a family of vortex line segments L_t along which maximum vorticity is comparable to $\Omega(t)$. Denote by $L(t)$ the arc length of L_t, $U_\xi(t) \equiv \max_{x,y \in L_t} |(u \cdot \xi)(x,t) - (u \cdot \xi)(y,t)|$, $U_n(t) \equiv \max_{L_t} |u \cdot n|$, and $M(t) \equiv \max(\|\nabla \cdot \xi\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)})$ where κ is the curvature of the vortex line and n is the unit normal vector of L_t. Further, we denote by $X(\alpha, t)$ the Lagrangian flow map [4].

Now we can state our second theorem.

Theorem 2. *Assume there is a family of vortex line segments L_t and $T_0 \in [0, T)$, such that $X(L_{t_1}, t_2) \supseteq L_{t_2}$ for all $T_0 < t_1 < t_2 < T$. We also assume that $\Omega(t)$ is monotonely increasing and $\|\omega(t)\|_{L^\infty(L_t)} \geq c_0 \Omega(t)$ for some $c_0 > 0$ when t is sufficiently close to T. Furthermore, we assume that*

1. $[U_\xi(t) + U_n(t)M(t)L(t)] \lesssim (T-t)^{-\alpha}$ *for some* $\alpha \in (0,1)$,
2. $M(t)L(t) \leq C_0$, *and*
3. $L(t) \gtrsim (T-t)^\beta$ *for some* $\beta < 1 - \alpha$,

then there will be no blow-up in the 3D incompressible Euler flow up to time T.

The proof of Theorem 2 relies on the geometric property of the 3-D Euler equation in a crucial way and will be deferred to Section 3. Here we would like to make a few remarks on the assumptions of the theorem and discuss how one can use this result to check the validity of some alleged 3-D Euler singularities obtained by numerical computations.

First, we remark that the first two assumptions of Theorem 2 are quite natural. From numerical computations, it has been observed that incompressible flows at later times are dominated by small regions of large magnitude of vorticity that shrink in

all three directions in the Eulerian coordinates. In fact, they should shrink in the Lagrangian coordinates as well. To see this, we argue that if they don't shrink in the Lagrangian coordinates, the volumes of these small regions would be non-decreasing since any Lagrangian region carried by the flow must maintain its volume due to the incompressibility of the flow. Thus these small regions must have at least one stretching direction along which the small regions grow in the Eulerian coordinates. This contradicts with the observation that these small regions shrink in all three directions. Now that these small regions shrink in all directions in the Lagrangian coordinates, it is reasonable to assume that there is one Lagrangian point $X(\alpha, t)$ that is contained in all these regions. Now if we take L_t to be the vortex line segment that passes $X(\alpha, t)$, then these two assumptions would be satisfied. Note that the assumption $M(t)L(t) \leq C$ is a sufficient condition to satisfy (2.1).

Next, we note that in Theorem 2, we used $U_\xi(t) + U_n(t)M(t)L(t)$ instead of the more observable quantity $U(t) \equiv \max_{L_t} |u(\cdot, t)|$. This is because we believe that $U_\xi(t) + U_n(t)M(t)L(t)$ may grow slower than $U(t)$. To see this, we first consider the term $U_\xi(t)$, which is defined as the maximum of the difference between the tangential velocity at any two points on L_t. In the case of collapsing vortex tubes, it is likely that the the tangential velocity has a better regularity along vortex lines than along the direction normal to vortex lines. In this case, the term $U_\xi(t)$ can be much smaller than the velocity itself. Even if such regularity is not available, we can bound this term by $2\max_{L_t} |u \cdot \xi|$. By the Biot-Savart law, we have

$$u \cdot \xi(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x + y, t) \cdot \xi(x, t)\ dy.$$

If vorticity is concentrated in a small region around x, and ξ has some regularity within this small region, then there will be an extra order of cancellation at $y = 0$ in the integral kernel for the tangential velocity. Therefore we should expect $u \cdot \xi$ to be smaller than $|u|$.

We now consider the term $U_n(t)M(t)L(t)$. In a regular vortex tube, if the maximum vorticity is achieved at the center vortex line, then U_n should correspond to the velocity of the motion of the vortex tube in the direction normal to itself. Even in the case of the vortex tube collapsing, the speed of the motion of the vortex tube itself is usually bounded. In this case, the maximum velocity component should come from the rotational component of the velocity. A formal argument based on Kelvin's circulation theorem shows that the rotational velocity component is of the order $\Omega(t)^{1/2}$. More generally, we can estimate the maximum velocity in terms of the maximum vorticity as follows:

$$U(t) \lesssim \Omega(t)^{3/5}, \tag{2.3}$$

where $U(t)$ and $\Omega(t)$ are the maximum velocity and maximum vorticity in the whole space respectively. This estimate will be proved rigorously in Lemma 4 in the Appendix

without making any regularity assumption on the vorticity vector. Therefore, as long as $\Omega(t) \lesssim (T-t)^{-5/3+\epsilon}$ for arbitrary small $\epsilon > 0$, then we will have $U(t) \lesssim (T-t)^{-1+\epsilon}$. In fact, in almost all numerical computations so far, $\Omega(t) \lesssim (T-t)^{-1}$. Thus, we can reasonably expect that $U(t) \lesssim (T-t)^{-3/5}$.

Theorem 2 can be viewed as a refinement of some existing theoretical results, and it is also more applicable to numerical observations. We illustrate these points through the following examples.

Example 2. First we review the theorem by Constantin-Fefferman-Majda [6]. In [6], Constantin, Fefferman, and Majda prove that no finite time blow-up can occur under two main assumptions: (i) $\int_0^T \|\nabla\xi\|^2_{L^\infty(W_t)}\, dt < \infty$, where $W_t \equiv X(W_0, t)$ with W_0 being an $O(1)$ Lagrangian region at $t = 0$; and (ii) $\|u(\cdot,t)\|_{L^\infty(W_t)}$ is bounded by some absolute constant U (For technical reasons, they also make some additional assumptions). Now if we further assume that $\|\nabla\xi\|_{L^\infty(W_t)}$ has a growth rate at $t \to T$, then their two main assumptions turn into

$$M(t) \ll (T-t)^{-1/2}, \qquad U_\xi(t) + U_n(t) \lesssim 2U.$$

Since W_t is carried by the flow, we can take L_t to be any vortex line in W_t. In particular, we can take L_t such that $L(t) \sim (T-t)^{1/2}$. This is equivalent to taking $\alpha = 0, \beta = 1/2$ in Theorem 2. We see that the three conditions in Theorem 2 are all satisfied and there will be no finite time blow-up.

Theorem 2 to some extent improves the previous result obtained by Constantin-Fefferman-Majda [6]. First of all, our result requires a weaker and very local assumption on the regularity of the vorticity vector ξ along one vortex line segment. In [6], the maximum norm of the gradient of the vorticity vector is assumed to be L^2 integrable in time in an $O(1)$ region containing the maximum vorticity, and the maximum norm of the velocity field is required to be bounded in this $O(1)$ region. In contrast, we essentially assume that the divergence of the vorticity vector and the curvature are integrable along a local vortex line segment whose length can shrink to zero as the time approaches to the alleged singularity time. The fact that the size of this local weakly regularly oriented region can shrink to zero with appropriate rate enables us to essentially eliminate the gap between our theoretical result and what has been observed numerically, which is significant.

Example 3. Next we review the main result by Cordoba-Fefferman [7]. There they consider a fixed cube region $Q \equiv I_1 \times I_2 \times I_3$, and a vortex tube Ω_t that only intersects with ∂Q twice at any time t, one in the upper face, and one in the lower face. Furthermore, $\|u(\cdot,t)\|_{L^\infty}$ is assumed to be integrable from 0 to T. Under these assumptions, they prove that the volume of Ω_t can not shrink to 0 as $t \nearrow T$.

If we assume that $M(t)$ is bounded by some constant, and $U(t)$ has a algebraic growth rate as $t \nearrow T$, then we can get the result in [7] by Theorem 2 by taking

$L(t) \sim 1$. Note that we can take $L(t) \sim 1$ since the cube region $I_1 \times I_2 \times I_3$ is fixed. In fact, in this case Theorem 2 rules out general blow-ups, while the result in [7] only rules out the possibility that the whole vortex tube shrink to one filament.

Example 4. Finally we apply our result to the numerical computations by Kerr. In a sequence of papers [9, 10, 11, 12], Kerr observed that when t is close enough to the alleged blow-up time T, the region bounded by the contour of $0.6\times$maximum vorticity has the length scale $(T-t)^{1/2}$ in the vortex line direction, and is contained in a box with length scale $(T-t)^{1/2}$ in all three directions. Within this region, vortex lines are "relatively straight" ([11]), except that in a smaller inner region they have curvature $\sim (T-t)^{-1/2}$ ([10]). It is also observed that the maximum velocity inside this region remains bounded up to time T([11]). Thus we can take $L(t) \sim (T-t)^{1/2}$, $M(t) \lesssim (T-t)^{-1/2}$, and $U_\xi(t), U_n(t) \sim (T-t)^0$. The assumptions in Theorem 2 are satisfied. Therefore it is quite possible that there will be no blow-up in this case. We plan to perform a more careful numerical study to further investigate this possible blow-up scenario by verifying the above scaling of various geometric and flow properties.

3 Proofs of Theorem 1 and Theorem 2

In this section, we prove our two main theorems in this paper. We first prove Theorem 1. Before we present the proof of Theorem 1, we first study the properties of the vorticity field. The key to our analysis is the incompressibility condition. It turns out that, when combined with the geometrical properties of the vorticity field, this condition becomes a constraint on the behavior of the flow, thus an obstacle for a finite time blow-up to occur.

3.1 Direction and magnitude of vorticity

It has long been observed that at later times incompressible flows are dominated by small vortex tubes in which the vorticity concentrates. This phenomenon has also been observed in recent numerical computations (Kerr [9, 10, 11, 12], Pelz [14, 15]). A vortex tube is a collection of vortex lines, so it is natural to study the behavior of the magnitudes of vorticity along one vortex line.

First, we have the following lemma, which relates, through the incompressibility condition, the vortex line geometry to the magnitude of vorticity.

Lemma 1. *Let $\xi(x,t) \stackrel{def}{=} \frac{\omega(x,t)}{|\omega(x,t)|}$ be the direction of the vorticity vector. Assume at a fixed time $t > 0$ the vorticity $\omega(x,t)$ is C^1 in x. We denote*

$$N = \{x \in R^3 : \omega(x,t) \neq 0\}.$$

Then at this time t, for any $x \in N$ there holds

$$\frac{\partial |\omega|}{\partial s}(x,t) = -(\nabla \cdot \xi(x,t)) |\omega| (x,t). \tag{3.1}$$

where s is the arc length variable along the vortex line passing x. Further more for any y that is on the same vortex line segment as x, (3.1) then gives

$$|\omega(y,t)| = |\omega(x,t)| \cdot e^{\int_x^y (-\nabla\cdot\xi)(s,t)ds}, \tag{3.2}$$

as long as the vortex line segment connecting x and y lies in N, where the integration is along the vortex line.

Proof. Notice that $\omega = |\omega| \xi$. Since $\omega(x,t) \neq 0$, $\xi \equiv \frac{\omega}{|\omega|}$ is well defined in a neighborhood of x. The incompressibility condition $\nabla \cdot \omega = 0$ then gives

$$\begin{aligned} 0 = \nabla \cdot \omega &= \nabla \cdot (|\omega| \xi) \\ &= (\nabla |\omega|) \cdot \xi + (\nabla \cdot \xi) |\omega| \\ &= (\xi \cdot \nabla) |\omega| + (\nabla \cdot \xi) |\omega| . \end{aligned} \tag{3.3}$$

It is easy to check that the directional derivative $\xi \cdot \nabla$ is actually the arc length derivative along the vortex line, i.e. $\xi \cdot \nabla = \frac{\partial}{\partial s}$. Therefore we obtain from (3.3) that

$$\frac{\partial |\omega|}{\partial s} = -(\nabla \cdot \xi) |\omega| , \tag{3.4}$$

with s being the arc length variable. Equation (3.2) then follows from integrating (3.4) along the vortex line. $\square$

Now we are ready to present a simple proof of Theorem 1.

Proof of Theorem 1: Using (3.2) and the assumption that $\int_0^T |\omega(y(t),t)| \ dt < \infty$ we obtain

$$\int_0^T |\omega(x(t),t)| \ dt < \infty.$$

Then by our assumption on $\omega(x(t),t)$, we have

$$\int_0^T \Omega(t) \ dt \lesssim \int_0^T |\omega(x(t),t)| \ dt < \infty.$$

Thus the theorem follows from the Beale-Kato-Majda theorem [2]. The estimate on the ratio of $\omega(x,t)$ and $\omega(y,t)$ is a direct consequence of (3.2). This completes the proof of Theorem 1.

3.2 Stretching of Vortex Lines

Before we prove Theorem 2, we need to study how the relative rate of arc length stretching along a vortex filament is related to the relative rate of maximum vorticity growth in time.

For any starting time t_1 and some time $t > t_1$, consider the evolution of a vortex line. Let s and β be the arc length parameter of this vortex line at time t and t_1 respectively. We can write, for this very vortex line, $s = s(\beta, t)$. Note that $s(\beta, t_1) = \beta$. Then we have the following lemma.

Lemma 2. *For any point α at time t_1 such that $\omega(\alpha, t_1) \neq 0$, let $X(\alpha, t)$ be the position of the same particle at time $t \geq t_1$. Then we have*

$$\frac{\partial s}{\partial \beta}(X(\alpha, t), t) = \frac{|\omega(X(\alpha, t), t)|}{|\omega(\alpha, t_1)|}. \tag{3.5}$$

Proof. Note that in our notation, $X(\alpha, t_1) = \alpha$. It is well known that for 3-D Euler flows we have [4]

$$\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) \cdot \omega(\alpha, t_1).$$

Then we obtain

$$\begin{aligned} |\omega(X(\alpha, t), t)| &= \xi(X(\alpha, t), t) \cdot \omega(X(\alpha, t), t) \\ &= \xi(X(\alpha, t), t) \cdot \nabla_\alpha X(\alpha, t) \cdot \xi(\alpha, t_1) \left|\omega(\alpha, t_1)\right|. \end{aligned}$$

Note that $\xi(X(\alpha, t), t) = \frac{\partial X(\alpha, t)}{\partial s}$ for any t, where s is the arc length variable of the vortex line that passes $X(\alpha, t)$ at time t. We can further simplify the above equations as

$$\begin{aligned} |\omega(X(\alpha, t), t)| &= \frac{\partial X(\alpha, t)}{\partial s} \cdot \nabla_\alpha X(\alpha, t) \cdot \frac{\partial \alpha}{\partial \beta} \left|\omega(\alpha, t_1)\right| \\ &= \frac{\partial X(\alpha, t)}{\partial s} \cdot \frac{\partial X(\alpha, t)}{\partial \beta} \left|\omega(\alpha, t_1)\right| \\ &= (\frac{\partial X(\alpha, t)}{\partial s} \cdot \frac{\partial X(\alpha, t)}{\partial s}) \frac{\partial s}{\partial \beta} \left|\omega(\alpha, t_1)\right| \\ &= \left|\xi(X(\alpha, t), t)\right|^2 \frac{\partial s}{\partial \beta} \left|\omega(\alpha, t_1)\right| \\ &= \frac{\partial s}{\partial \beta} \left|\omega(\alpha, t_1)\right|. \end{aligned}$$

This completes the proof of Lemma 2. ∎

It is well-known that the evolution of the magnitude of vorticity along any particle path is governed by the following equation ([5]),

$$\frac{D\,|\omega(x,t)|}{Dt} = \xi(x,t)\cdot(\nabla u(x,t)\cdot\xi(x,t))\,|\omega(x,t)|\,, \tag{3.6}$$

where $\frac{D}{Dt} \equiv \partial_t + u\cdot\nabla$ is the material derivative. Then the above lemma immediately gives the equation that governs the arc length stretching s_β. If we denote $x = X(\alpha,t)$, then we have

$$\begin{aligned}\frac{Ds_\beta}{Dt}(x,t)) &= [\xi\cdot(\nabla u\cdot\xi)]\,s_\beta \\ &= [(\xi\cdot\nabla u)\cdot\xi]\,s_\beta \\ &= [(\xi\cdot\nabla)(u\cdot\xi) - u\cdot(\xi\cdot\nabla)\xi]\,s_\beta \\ &= [(u\cdot\xi)_s - \kappa(u\cdot n)]\,s_\beta \\ &= (u\cdot\xi)_\beta - \kappa(u\cdot n)s_\beta,\end{aligned} \tag{3.7}$$

where we have used the fact $\xi\cdot\nabla = \frac{\partial}{\partial s}$ and the well-known basic relation in differential geometry

$$\frac{\partial\xi}{\partial s} = \kappa n, \tag{3.8}$$

with $\kappa = |\xi\cdot\nabla\xi|$ being the curvature and n the unit normal vector of the vortex line.

Now we integrate equation (3.7) along the vortex line:

$$\begin{aligned}\frac{D[s(\beta_2,t) - s(\beta_1,t)]}{Dt} &= (u\cdot\xi)(X(\beta_2,t),t) - (u\cdot\xi)(X(\beta_1,t),t) \\ &\quad - \int_{\beta_1}^{\beta_2} \kappa(X(\eta,t),t)\cdot(u\cdot n)s_\eta\, d\eta.\end{aligned} \tag{3.9}$$

Further, we integrate (3.9) over some time interval $[t_1, t]$. We get

$$\begin{aligned}s(\beta_2,t) - s(\beta_1,t) &= s(\beta_2,t_1) - s(\beta_1,t_1) \\ &\quad + \int_{t_1}^{t} ((u\cdot\xi)(X(\beta_2,\tau),\tau) - (u\cdot\xi)(X(\beta_1,\tau),\tau))\ d\tau \\ &\quad - \int_{t_1}^{t}\int_{\beta_1}^{\beta_2} \kappa(\eta,\tau)\cdot(u\cdot n)s_\eta\, d\eta d\tau.\end{aligned} \tag{3.10}$$

Let $l(t) \equiv s(\beta_2,t) - s(\beta_1,t) > 0$ and denote by l_{12} the vortex line segment connecting the points $X(\beta_1,t)$ and $X(\beta_2,t)$. It follows from (3.10) that

$$
\begin{aligned}
l(t) \quad &\le \quad l(t_1) + \int_{t_1}^{t} (|(u\cdot\xi)(X(\beta_2,\tau),\tau) - (u\cdot\xi)(X(\beta_1,\tau),\tau)|)d\tau \\
&\quad + \int_{t_1}^{t} M(\tau)\|u\cdot n\|_{L^\infty(l_{12})}(\tau)l(\tau)\ d\tau,
\end{aligned} \tag{3.11}
$$

where $M(t) = \max(\|\nabla\cdot\xi\|_{L^\infty(l_{12})}, \|\kappa\|_{L^\infty(l_{12})})$. Inequality (3.11) reveals how the stretching of vortex lines is controlled by the velocity field and the geometry of the vorticity field. Furthermore, we will derive an inequality to bound the relative ratio of the magnitudes of vorticity at different time by the relative ratio of the arc lengths of the vortex lines. This provides a sharp estimate on the growth rate of vorticity in terms of the arc length stretching of vortex lines.

Lemma 3. *Let l_t be a vortex line segment that is carried by the flow. Denote its length by $l(t)$, and let $M(t)$ be defined as in Theorem 2. Then for any point $X(\alpha',t)\in l_t$, we have*

$$
e^{-(M(t)l(t)+M(t_1)l(t_1))}\frac{|\omega(X(\alpha',t),t)|}{|\omega(\alpha',t_1)|} \le \frac{l(t)}{l(t_1)} \le e^{(M(t)l(t)+M(t_1)l(t_1))}\frac{|\omega(X(\alpha',t),t)|}{|\omega(\alpha',t_1)|}. \tag{3.12}
$$

Proof. Let β denote the arc length parameter at time t_1. Denote by l_t the vortex line segment from 0 to β, and use s as the arc length parameter at time t. Now by the mean value theorem and Lemma 2 we have

$$
\frac{l(t)}{l(t_1)} = \frac{\int_0^\beta s_\beta(\eta)\ d\eta}{\beta} = s_\beta(\eta') = \frac{|\omega(X(\alpha'',t),t)|}{|\omega(\alpha'',t_1)|},
$$

for some α'' on the same vortex line. Now (3.12) follows from Lemma 1. This completes the proof of Lemma 3. $\square$

By combining (3.12) and (3.11), we obtain

$$
\begin{aligned}
|\omega(X(\alpha,t),t)| \quad &\le \quad e^{(M(t)l(t)+M(t_1)l(t_1))}|\omega(X(\alpha,t_1),t_1)| \\
&\quad \cdot\left[1 + \frac{C}{l(t_1)}\int_{t_1}^{t}(U_\xi(\tau) + M(\tau)U_n(\tau)l(\tau))\ \ d\tau\right],
\end{aligned} \tag{3.13}
$$

for any $X(\alpha,t)$ that lies in l_t. Let $\Omega_l(t) = \|\omega(\cdot,t)\|_{L^\infty(l_t)}$. We can easily derive from (3.13) the following inequality:

$$
\Omega_l(t) \le e^{(M(t)l(t)+M(t_1)l(t_1))}\Omega_l(t_1)\left[1 + \frac{C}{l(t_1)}\int_{t_1}^{t}(U_\xi(\tau) + M(\tau)U_n(\tau)l(\tau))\,d\tau\right]. \tag{3.14}
$$

Inequality (3.14) shows how the growth of vorticity is controlled by the properties of the flow. This inequality is important to our analysis in our proof of Theorem 2 and will be used heavily.

3.3 Interplay between the Geometry and Growth Rate

This subsection is devoted to the proof of Theorem 2.

Proof of Theorem 2: We prove Theorem 2 by contradiction. First, by translating the initial time we can assume that the assumptions in Theorem 2 hold in $[0,T)$. Define

$$r \equiv (R/c_0) + 1, \tag{3.15}$$

where $R \equiv e^{2C_0}$, C_0 is the constant in the theorem such that $M(t)L(t) \leq C_0$ for all $t \in [0,T)$, and c_0 is the constant such that $\Omega_L(t) \geq c_0 \Omega(t)$. Throughout the proof we denote $\Omega_L(t) \equiv \|\omega(\cdot,t)\|_{L^\infty(L_t)}$. The reason for choosing the parameter r this way will become clear later in the proof. If there were a finite time blow-up at time T, we would have

$$\int_0^T \Omega(t)\; dt = \infty,$$

or equivalently for any $t_1 \in [0,T)$,

$$\int_{t_1}^T \Omega(t)\; dt = \infty.$$

Then necessarily we have $\Omega(t) \nearrow \infty$ as $t \nearrow T$. Now we can take a time sequence $t_1, t_2, \ldots, t_n, \ldots$ such that

$$\Omega(t_{k+1}) = r\Omega(t_k), \tag{3.16}$$

where r is defined in (3.15). Since $\Omega(t)$ is monotone, and T is the smallest time such that $\int_0^T \Omega(t)\; dt = \infty$, it is obvious that $t_n \nearrow T$ as $n \to \infty$.

Now we choose $l_{t_2} = L_{t_2}$. By our assumptions on L_t, there is $l_{t_1} \subset L_{t_1}$ such that $X(l_{t_1}, t_2) = l_{t_2}$. If we further denote

$$\Omega_l(t_i) \equiv \|\omega(\cdot,t)\|_{L^\infty(l_{t_i})} \quad i = 1,2,$$

then by taking $t = t_2$ in (3.12) we would have

$$\begin{aligned} l(t_1) \quad &\geq \quad l(t_2)\frac{1}{R}\frac{|\omega(\alpha', t_1)|}{|\omega(X(\alpha',t_2),t_2)|} \\ &\geq \quad l(t_2)\frac{1}{R^2}\frac{\Omega_L(t_1)}{\Omega_L(t_2)}, \end{aligned}$$

where the last inequality is due to the assumption $M(t)L(t) \leq C_0$ and Theorem 1. Note that by assumption we have $\Omega_L(t) \geq c_0\Omega(t)$. Thus $l(t_1)$ can be further bounded from below by

$$\begin{aligned} l(t_1) &\geq l(t_2)\frac{c_0}{R^2}\frac{\Omega(t_1)}{\Omega(t_2)} \\ &= Cl(t_2) = CL(t_2) \gtrsim (T-t_2)^{\beta}, \end{aligned} \tag{3.17}$$

where $C = \frac{c_0}{R^2 r}$ is independent of time.

On the other hand, we have from (3.14)

$$\Omega_l(t_2) \leq e^{(M(t_2)l(t_2)+M(t_1)l(t_1))}\Omega_l(t_1)\left[1+\frac{C}{l(t_1)}\int_{t_1}^{t_2}(U_\xi(\tau)+M(\tau)U_n(\tau)l(\tau))\ d\tau\right]. \tag{3.18}$$

By the assumption of Theorem 2, we have

$$\begin{aligned} M(t_2)l(t_2)+M(t_1)l(t_1) &\leq C_0 \\ U_\xi(\tau)+U_n(\tau)M(\tau)l(\tau) &\lesssim (T-\tau)^{-\alpha}. \end{aligned}$$

Then it follows from (3.18) and (3.17) that

$$\Omega_l(t_2) \leq R\Omega_l(t_1) + C\frac{\Omega_l(t_1)}{(T-t_2)^{\beta}}\int_{t_1}^{t_2}(T-\tau)^{-\alpha}\ d\tau.$$

Note that the constant C here depends on R, r and c_0.

Applying our assumption that $\Omega_L(t) \geq c_0\Omega(t)$, we have

$$\begin{aligned} \Omega(t_2) &\leq \frac{1}{c_0}\Omega_L(t_2) = \frac{1}{c_0}\Omega_l(t_2) \\ &\leq \frac{R}{c_0}\Omega_l(t_1) + \frac{C}{c_0}\frac{\Omega_l(t_1)}{(T-t_2)^{\beta}}\int_{t_1}^{t_2}(T-\tau)^{-\alpha}\ d\tau \\ &\leq \frac{R}{c_0}\Omega(t_1) + \frac{C}{c_0}\frac{\Omega(t_1)}{(T-t_2)^{\beta}}\int_{t_1}^{t_2}(T-\tau)^{-\alpha}\ d\tau \\ &\leq (r-1)\Omega(t_1) + \frac{C}{(1-\alpha)c_0}\frac{\Omega(t_1)}{(T-t_2)^{\beta}}\left[(T-t_1)^{1-\alpha}-(T-t_2)^{1-\alpha}\right], \end{aligned}$$

where $r = (R/c_0)+1$ is defined in (3.15). We still denote $C/(c_0(1-\alpha))$ by C. The generic constant C now depends on R, r, c_0 and is proportional to $(1-\alpha)^{-1}$. Since $(T-t_2)^{1-\alpha} > 0$, we can discard it and obtain

$$\Omega(t_2) \leq (r-1)\Omega(t_1) + C\Omega(t_1)\frac{(T-t_1)^{1-\alpha}}{(T-t_2)^{\beta}}. \tag{3.19}$$

Since $\Omega(t_2) = r\Omega(t_1)$, we can cancel $\Omega(t_1)$ from both sides of (3.19) and obtain

$$r \leq (r-1) + C\frac{(T-t_1)^{1-\alpha}}{(T-t_2)^{\beta}},$$

which gives

$$(T-t_2)^{\beta} \leq C(T-t_1)^{1-\alpha},$$

or equivalently

$$(T-t_2) \leq C(T-t_1)^{1+2\delta},$$

where

$$2\delta \equiv \frac{1-\alpha}{\beta} - 1.$$

Now it is quite clear that why we take $\Omega(t_2)/\Omega(t_1) = r > R/c_0$ and choose $r = (R/c_0) + 1$.

Now note that t_1 is independent of C and δ, so we can take t_1 close enough to T such that $C(T-t_1)^{\delta} < 1$. Then we have

$$(T-t_2) \leq (T-t_1)^{1+\delta}.$$

By doing the same thing to each pair (t_k, t_{k+1}), we get

$$(T-t_{k+1}) \leq (T-t_k)^{1+\delta} \leq (T-t_1)^{(1+\delta)^k}.$$

If we take $(T-t_1) < 1$, this reduces to

$$(T-t_{k+1}) \leq (T-t_1)^{1+k\delta} = (T-t_1)\left((T-t_1)^{\delta}\right)^k. \tag{3.20}$$

Now we study the condition of $\int_{t_1}^{T} \Omega(t)\, dt = \infty$ more carefully. By the assumption that $\Omega(t)$ is monotonely increasing, we have

$$\begin{aligned}
\int_{t_1}^{T} \Omega(t)\, dt &= \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} \Omega(t)\, dt \\
&\leq \sum_{k=1}^{\infty} \Omega(t_{k+1})(t_{k+1}-t_k) \\
&= \Omega(t_1) \sum_{k=1}^{\infty} r^k (t_{k+1}-t_k),
\end{aligned}$$

where we have used $\Omega(t_{k+1}) = r\Omega(t_k) = r^k\Omega(t_1)$. Since $\int_{t_1}^{T} \Omega(t)\ dt = \infty$, we obtain

$$\sum_{k=1}^{\infty} r^k (t_{k+1} - t_k) = \infty.$$

From this, we conclude that

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{l=0}^{k-1}(r^{l+1} - r^l)(t_{k+1} - t_k) &= \sum_{k=1}^{\infty}(r^k - 1)(t_{k+1} - t_k) \\
&= \sum_{k=1}^{\infty} r^k (t_{k+1} - t_k) - (T - t_1) \\
&= \infty.
\end{aligned}$$

Since $r = (R/c_0) + 1 > 1$, all the terms $(r^{l+1} - r^l)(t_{k+1} - t_k)$ in the summation are positive. We can exchange the order of the summation to get

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{l=0}^{k-1}(r^{l+1} - r^l)(t_{k+1} - t_k) &= \sum_{l=0}^{\infty}\sum_{k=l+1}^{\infty}(r^{l+1} - r^l)(t_{k+1} - t_k) \\
&= \sum_{l=0}^{\infty}(r^{l+1} - r^l)(T - t_{l+1}) \\
&= (r-1)\sum_{l=0}^{\infty} r^l (T - t_{l+1}),
\end{aligned}$$

which implies

$$\sum_{k=0}^{\infty} r^k (T - t_{k+1}) = \infty.$$

By substituting (3.20) into the above equation, we get

$$\sum_{k=0}^{\infty} \left[r(T - t_1)^{\delta} \right]^k = \infty. \tag{3.21}$$

Note that we can take t_1 arbitrarily close to T. In particular, we can take t_1 such that $r(T - t_1)^{\delta} < 1$. This implies $\sum_{k=0}^{\infty} \left[r(T - t_1)^{\delta} \right]^k < \infty$. This contradicts with (3.21). Therefore we conclude that $\int_0^T \Omega(t)dt < \infty$. It then follows from the Beale-Kato-Majda blow-up criterion that there will be no blow-up up to time T. This completes the proof of Theorem 2.

Appendix. Estimate of Maximum Velocity by Maximum Vorticity

In this appendix, we prove the following lemma:

Lemma 4. *Let $u(x,t)$ be the solution to 3-D Euler equations (1.1), and $\omega(x,t) \equiv \nabla \times u(x,t)$ be the vorticity. Denote $\Omega(t) \equiv \|\omega(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}$ and $U(t) \equiv \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}$. Then the following inequality holds:*

$$U(t) \lesssim \Omega(t)^{3/5}.$$

Proof. By the well-known Biot-Savart law [4], we have

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y,t)\ dy.$$

Take a common smooth cut-off function $\chi : \{0\} \cup \mathbb{R}^+ \mapsto [0,1]$ such that $\chi(r) = 1$ for $r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. Let $\rho > 0$ be a small positive parameter to be determined later. Then we have

$$\begin{aligned}
|u(x,t)| &= \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y,t)\ dy\right| \\
&= \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \chi(\frac{|y|}{\rho})\frac{y}{|y|^3} \times \omega(x+y,t)\ dy + \frac{1}{4\pi}\int_{\mathbb{R}^3}(1-\chi(\frac{|y|}{\rho}))\frac{y}{|y|^3} \times (\nabla \times u(x+y,t))\ dy\right|.
\end{aligned}$$

Invoking integration by part in the second integral, we have

$$\begin{aligned}
|u(x,t)| \quad \leq \quad & \frac{1}{4\pi}\Omega(t)\int_{|y|\leq 2\rho} \frac{1}{|y|^2}\ dy \\
& + C\int_{|y|\geq\rho} \frac{1}{|y|^3}\,|u(x+y,t)|\ dy \\
& + C'\frac{1}{\rho}\int_{|y|\geq\rho} \frac{1}{|y|^2}\,|u(x+y,t)|\ dy.
\end{aligned}$$

Using the polar coordinate in the first integral, and the Schwarz inequality in the other two, we obtain

$$|u(x,t)| \quad \leq \quad C\left[\Omega(t)\rho + \left(\int_{|y|\geq\rho} \frac{1}{|y|^6}\ dy\right)^{1/2} + \frac{1}{\rho}\left(\int_{|y|\geq\rho} \frac{1}{|y|^4}\ dy\right)^{1/2}\right],$$

where we have used the fact that $\|u\|_{L^2(\mathbf{R}^3)}$ is conserved in time [4], i.e. $\|u\|_{L^2(\mathbf{R}^3)} = \|u_0\|_{L^2(\mathbf{R}^3)} \leq C$.

Finally we use the polar coordinates in the last two integrals, and get

$$\begin{aligned} |u(x,t)| &\leq C\left[\Omega(t)\rho + \left(\int_\rho^\infty \frac{1}{r^4}\,dr\right)^{1/2} + \frac{1}{\rho}\left(\int_\rho^\infty \frac{1}{r^2}\,dr\right)^{1/2}\right] \\ &\leq C\left[\Omega(t)\rho + \rho^{-3/2}\right]. \end{aligned}$$

By taking $\rho = \Omega(t)^{-2/5}$, we prove the desired estimate. This completes the proof of Lemma 4.

$\square$

References

[1] A. Babin, A. Mahalov & B. Nicolaenko, *3D Navier-Stokes and Euler Equations with Initial Data Characterized by Uniformly Large Vorticity*, Indiana University Mathematics Journal, Vol. 50, No. 1, 1–35, 2001.

[2] J. T. Beale, T. Kato, A. Majda, *Remarks on the breakdown of smooth solutions for the* 3*-D Euler equations,* Commun.Math.Phys. Vol. 94, 61–66, 1984.

[3] R. E. Caflisch, *Singularity formation for complex solutions of the 3D incompressible Euler equations*, Physica D, 67, 1–18, 1993.

[4] A. J. Chorin & J. E. Marsden, **A Mathematical Introduction to Fluid Mechanics**, 3rd ed., Springer-Verlag, New York, 1993.

[5] P. Constantin, *Geometric Statistics in Turbulence*, SIAM Review, Vol. 36, No. 1, 73 – 98, March 1994.

[6] P. Constantin, C. Fefferman, A. Majda, *Geometric constraints on potentially singular solutions for the* 3*-D Euler equations*, Commun.P.D.E., 21, 559–571. 1996.

[7] D. Cordoba, C. Fefferman, *On the collapse of tubes carried by 3D incompressible flows*, Commun. Math. Phys. Vol. 222, No. 2, 293–298, 2001.

[8] R. Grauer, C. Marliani & K. Germaschewski, *Adaptive Mesh Refinement for Singular Solutions of the Incompressible Euler Equations*, Physical Review Letters, Vol. 80, No. 19, 11 May 1998.

[9] R. M. Kerr, *Evidence for a singularity of the three-dimensional incompressible Euler equations*, Phys. Fluids A, 5, 1725-1746, 1993.

[10] R. M. Kerr, *The role of singularities in Euler*, In **Small-scale structure in hydro and magnetohydrodynamic turbulence**, eds. A. Pouquet, P. L. Sulem, Lecture Notes, Springer-Verlag, 1995.

[11] R. M. Kerr, *Euler Singularities and Turbulence*, In 19th ICTAM Kyoto' 96, eds. T. Tatsumi, E. Watanabe and T. Kambe, Elsevier Science.

[12] R. M. Kerr, *The Outer Regions in Singular Euler*, In **Fundamental Problematic Issues in Turbulence**, eds. Tsinober and Gyr, Birkhäuser. 1998.

[13] T. Ogawa, Y. Taniuchi, *On blow up criteria of smooth solutions to the 3-D Euler equations in a bounded domain*, JDE, Vol. 190, No. 1, 39–63, May 1 2003.

[14] R. B. Pelz, *Symmetry and the hydrodynamic blow-up problem*, J Fluid Mech., 444, 299–320, Oct. 2001.

[15] R. B. Pelz, *Locally self-similar, finite-time collapse in a high-symmetry vortex filament model*, Physical Review E, Vol.55, No.2, 1617–1620, Feb. 1997.